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Non-uniform slot injection into a laminar boundary layer

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SUMMARY

Slot injection into a laminar boundary layer in both supersonic and subsonic flow is considered. The blowing rates are sufficiently large to provoke an interaction between the boundary layer and outer inviscid flow, and this interaction is accounted for by triple-deck theory. The non-uniform nature of the blowing velocity models the channel flow from which the injection takes place.

1. Introduction

In this paper we consider non-uniform blowing from a narrow slot in the boundary into a two-dimensional laminar boundary layer in either a subsonic or supersonic flow. The injected fluid emerges obliquely from a channel, and the non-uniformity is associated with the assumed velocity profile in the channel. The blowing is supposed sufficiently strong to induce an interaction between the boundary layer and oncoming stream and in order to describe this interaction triple-deck scales are assumed (see for example Stewartson [1]). Thus if Re is a representative Reynolds number then, in dimensionless form, we choose the slot width to be $O(Re^{-3/8})$ and the tangential and normal components of the blowing velocity to be $O(Re^{-1/8})$, $O(Re^{-3/8})$ respectively.

In triple-deck theory the key feature, which distinguishes it from the clearly defined hierarchy of classical boundary-layer theory, is that the pressure gradient which controls the flow within the lower deck is itself generated by changes in the displacement thickness of the boundary layer with which it must be simultaneously calculated.

Practical reasons for blowing into boundary layers include control of the boundary layer and a means for effecting a reduction of heat transfer across the boundary as, for example, on a turbine blade. Early work on blowing into boundary layers was restricted to weak blowing rates in which the normal velocity is $O(Re^{-1/2})$ (see for example Pretsch [2], Acrivos [3], Watson [4]). More recently strong blowing, within the same triple-deck framework as is adopted here, has been considered by Smith and Stewartson [5], and Napolitano and Messick [6] who consider uniform normal injection into a boundary layer in supersonic flow and subsonic flow respectively, and Riley [7] who considers uniform oblique injection into a boundary layer in a supersonic flow. For the case of uniform normal injection there is a finite discontinuity in the pressure gradient at each of the leading and trailing edges of the slot. The introduction of a uniform tangential velocity in the slot, as for oblique injection, results in an infinite pressure gradient at

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the slot edges, where the shear stress also becomes unbounded. One of the aims of the present work is to introduce a more realistic blowing velocity distribution into the slot as would arise, for example, when the fluid emerges from a channel below the boundary. As we shall see, when the velocity at the boundary is continuous in oblique injection, the pressure gradient and shear stress are continuous, although the latter has an infinite gradient at the slot edges whose removal requires a study on a scale even smaller than that of the triple deck.

In a supersonic flow the displacement-induced pressure is determined according to Ackeret's law, and as a consequence in the downstream integration the pressure may be determined at each step. There is one free parameter in the problem, for example the slot length, which is adjusted to enable conditions well downstream from the slot to be satisfied. In the examples which we give we choose a slot length which is the same as in one of the examples considered by Riley [7], with the same pressure rise at the leading edge of the slot following a free interaction ahead of the slot. In a subsonic flow the pressure/displacement relation is more complicated and is represented by a Hilbert integral. Earlier subsonic triple-deck studies have involved overall iteration with substantial under-relaxation to avoid divergence, as for example in the study by Jobe and Burggraf [8] of the flow at the trailing edge of a flat plate, or have involved the solution of an unsteady problem as in Napolitano, Werle and Davis [9], and Napolitano and Messick [6]. A further aim of the present paper is to demonstrate the effectiveness of a method recently introduced by Veldman [10] for problems of this type in which no under-relaxation is required and which renders the calculation of subsonic triple-decks almost as easy as those in the corresponding supersonic case. In the examples we give we adopt the same geometrical configuration as for the supersonic case, and it is seen that qualitatively the overall flow features are the same for both sub- and supersonic flow.

2. Problem formulation

We suppose that fluid of viscosity $\bar{\mu}$ and thermal conductivity $\bar{k}$ flows with uniform speed $\bar{U}_\infty$ past a flat plate of length L which is maintained at a uniform temperature $\bar{T}_w$. At a distance $\bar{x}_s$ from the leading edge fluid is injected into the oncoming stream from a narrow slot. We define

$$Re = \epsilon^{-8} = \frac{\bar{U}_\infty L}{\bar{\nu}_\infty}, \tag{2.1}$$

where $\bar{\nu}$ is the kinematic viscosity of the fluid, as the Reynolds number, and we assume $Re \gg 1$. The length and velocity scales associated with the slot blowing are chosen to accommodate the triple-deck theory of Stewartson and Messiter (see for example Stewartson [1]). Thus the slot is assumed to be of width $O(\epsilon^3 L)$ and the injected fluid emerges from it with speed $O(\epsilon\, \bar{U}_\infty)$ inclined at an angle $O(\epsilon^2)$ to the oncoming stream.

Through the triple-deck structure we are able to describe completely, for this example of strong blowing, the interaction between the boundary layer and free stream, and in particular show how the pressure field anticipates the slot upstream and readjusts downstream from it. In the theory of the triple deck there is a main deck, of classical boundary-layer thickness $O(\epsilon^4 L)$, in which the flow behaves passively but because of the thickening of the lower deck provides a displacement effect, which is responsible for the induced pressure variation. The displacement-pressure relationship is determined in an inviscid upper layer of thickness $O(\epsilon^3 L)$. The flow in the lower deck, of thickness $O(\epsilon^5 L)$, is ultimately responsible for the displacement effect, but we again emphasize that in the solution procedure the pressure and displacement are calculated simultaneously. This contrasts sharply with the hierarchical procedure of classical boundary-layer theory. Thus in the lower deck we write

$$\bar{x} = \bar{x}_s + \bar{x}_s \frac{\bar{C}^{3/8}(\bar{T}_w/\bar{T}_\infty)^{3/2}}{\alpha^{5/4}\,|M_\infty^2-1|^{3/8}}\,\epsilon^3 X, \qquad \bar{y} = \bar{x}_s \frac{\bar{C}^{5/8}(\bar{T}_w/\bar{T}_\infty)^{3/2}}{\alpha^{3/4}\,|M_\infty^2-1|^{1/8}}\,\epsilon^5 Y,$$

$$\bar{u} = \bar{U}_\infty \frac{\alpha^{1/4}\bar{C}^{1/8}(\bar{T}_w/\bar{T}_\infty)^{1/2}}{|M_\infty^2-1|^{1/8}}\,\epsilon U(X,Y), \qquad \bar{v} = \bar{U}_\infty \alpha^{3/4}\,|M_\infty^2-1|^{1/8}\,\bar{C}^{3/8} \times (\bar{T}_w/\bar{T}_\infty)^{1/2}\epsilon^3 V(X,Y), \tag{2.2}$$

$$\frac{\bar{p}-\bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2} = \bar{C}^{1/4}\alpha^{1/2}\,|M_\infty^2-1|^{-1/4}\epsilon^2 P(X),$$

where $(\bar{x}, \bar{y})$ are co-ordinates measured along and normal to the boundary; the Chapman viscosity law is assumed so that $\bar{C} = \bar{\mu}_w \bar{T}_\infty / \bar{\mu}_\infty \bar{T}_w$, and $\alpha = 0.332 \ldots$ is determined from the oncoming boundary-layer flow. We then have, as the fundamental problem of the triple-deck theory

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{dP}{dX} + \frac{\partial^2 U}{\partial Y^2}, \quad \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{2.3a, b}$$

with boundary conditions on $Y = 0$,

$$U = V = 0, \quad X < 0 \quad \text{and} \quad X > \ell; \quad U = U_w(X), \quad V = V_w(X), \quad 0 < X < \ell, \tag{2.4}$$

together with the conditions

$$U - Y \to 0 \quad \text{as} \quad X \to -\infty,$$

$$\text{and} \quad U - Y \to A(X) \quad \text{as} \quad Y \to \infty, \tag{2.5}$$

which allow us to match our solution with the flow upstream, and with the main deck respectively. The displacement function $A(X)$ is related to the pressure $P(X)$ following a consideration of the flow in the upper deck. Thus we have, for supersonic flow

$$P(X) = -A'(X), \tag{2.6a}$$

where a prime denotes differentiation with respect to X, whilst for subsonic flow the corresponding relationship is

$$P(X) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A'(\xi)}{\xi - X}\,d\xi. \tag{2.6b}$$

The prescribed velocities U_w, V_w on $0 < X < \ell$ are themselves determined from an oncoming flow. Thus we assume that the injected fluid emerges from a channel, as in Figure 1, in which there is a parallel flow whose profile $W(s)$ is given by

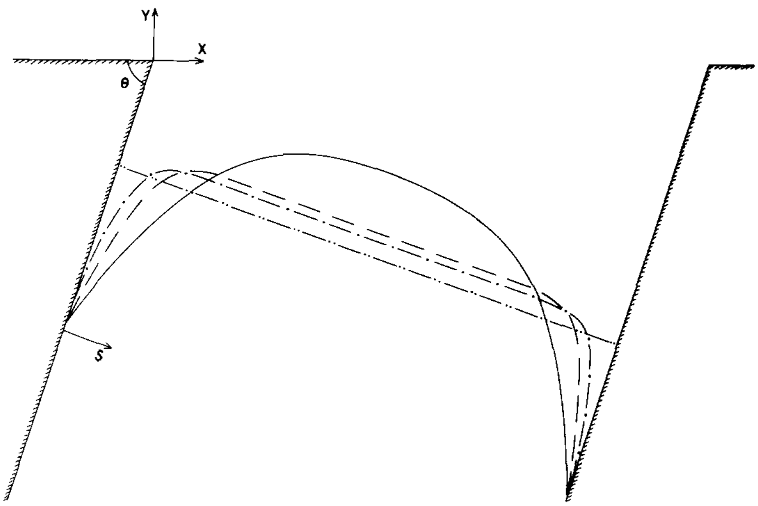

Figure 1. The slot configuration. The slot length ℓ = 5·563 and the channel inclination $\theta = \tan^{-1} 3$. With reference to equation (2.7) the channel velocity profiles correspond to the pairs of values (W_s, λ) as follows ——— (0.9789,1), - - - - - (0.7340, 3), —·—·— (0.6917, 5), —··—··— (0.6325, ∞).

$$W = W_s\,\{\tanh\lambda s + \tanh\lambda(\ell_s - s) - \tanh\lambda\ell_s\}, \tag{2.7}$$

where s is a co-ordinate measured across the channel, and ℓ_s is the channel width in our scaled co-ordinates. Varying the parameter λ enables us to model a channel flow which varies from a fully-developed form to one in which boundary layers surround a core of uniform speed. Figure 1 illustrates this by showing ch' ınel profiles for various values of λ. This form for the assumed slot velocity is more realistic than the uniform values, with their attendant discontinuities, adopted by Riley [7].

In the next section we discuss our method of solution of the problem posed by equations (2.3) to (2.6) treating the cases of supersonic and subsonic flow separately.

3. Solution procedure

Although the supersonic and subsonic cases must necessarily be treated separately, on account of the different pressure-displacement relations (2.6a), (2.6b), our method for the numerical integration of the boundary-layer equations (2.3) is the same in each case. This is based upon the original treatment of Stewartson and Williams [11] and is implemented as follows.

It is convenient to eliminate the pressure from (2.3a) and work with the two-dimensional stream function Ψ which satisfies, from (2.3)

$$\frac{\partial^4 \Psi}{\partial Y^4} + \frac{\partial \Psi}{\partial X}\frac{\partial^3 \Psi}{\partial Y^3} - \frac{\partial \Psi}{\partial Y}\frac{\partial^3 \Psi}{\partial X \partial Y^2} = 0, \tag{3.1}$$

together with, from (2.5),

$$\frac{\partial \Psi}{\partial Y} \to Y \quad \text{as} \quad X \to -\infty, \qquad \frac{\partial^2 \Psi}{\partial Y^2} \to 1 \quad \text{as} \quad Y \to \infty, \tag{3.2a, b}$$

and from (2.4)

$$\begin{aligned} &\Psi = \frac{\partial \Psi}{\partial Y} = 0, \quad X < 0, \quad Y = 0;\\ &\Psi = -\int_0^X V_w(X)\,\mathrm{d}X, \quad \frac{\partial \Psi}{\partial Y} = U_w(X), \quad 0 < X < \ell, \quad Y = 0;\\ &\Psi = -\int_0^\ell V_w(X)\,\mathrm{d}X, \quad \frac{\partial \Psi}{\partial Y} = 0, \quad X > \ell, \quad Y = 0. \end{aligned} \tag{3.3}$$

An additional boundary condition is required for (3.1) and this is derived from an evaluation of (2.3a) at $Y = 0$ together with (2.5b) and either (2.6a) or (2.6b) depending upon the case under consideration. For the moment we write this condition simply as

$$F(\Psi) = 0. \tag{3.4}$$

By initiating the solution in an appropriate manner at a sufficiently large negative value of X we advance the solution step by step as follows. Equation (3.1) is first discretized in the streamwise direction so that at each step we have to solve a nonlinear ordinary differential equation which is quasi-linearized and solved iteratively. This is achieved by first introducing a new independent variable Z defined by

$$Z = \frac{2Y - Y_\infty}{Y_\infty}, \tag{3.5}$$

where Y_∞ is a finite quantity chosen to be sufficiently large to model the outer edge of the lower deck. The solution is then represented as a finite series of Chebychev polynomials, thus

$$\Psi = \sum_{i=0}^{N} a_i T_i(Z), \tag{3.6}$$

where $a_i (i = 0, 1, ..., N)$ are $N + 1$ unknown coefficients and T_i is the Chebychev polynomial of degree i. The coefficients are determined by satisfying the equation to be solved, exactly, at the $N - 3$ 'selected points'

$$Z_r = \cos\{r\pi/(N-4)\}, \quad r = 0, 1, ..., N-4, \tag{3.7}$$

and satisfying the four boundary conditions given by (3.2b), (3.3) and (3.4) to give $N + 1$ equations for the unknowns a_i. The disadvantages of this method are associated with the time-penalty incurred in the inversion of a matrix with no particular structure that can be exploited. The advantages of the method lie with the rapid convergence of the series (3.6), together with the fact that since $|T_i(Z)| \leqslant 1$ the last coefficient a_N provides a ready estimate of the accuracy of the solution. Also, since the selected points (3.7) tend to be more densely distributed at each end of the range the method is well-suited to problems of a multi-structured type when a region of rapid change is close to one end of the range.

We now consider the application of the above to each of the supersonic and subsonic flows in turn.

(i) *Supersonic flow*

The form of $F(\Psi)$ in (3.4) is determined as follows. From (2.5b) and (2.6a) we have

$$\left.\frac{\partial^2 \Psi}{\partial X \partial Y}\right|_{Y=Y_\infty} = -P(X), \tag{3.8}$$

and from (2.3a)

$$P'(X) = \left.\frac{\partial^3 \Psi}{\partial Y^3}\right|_{Y=0} - V_w \left.\frac{\partial^2 \Psi}{\partial Y^2}\right|_{Y=0} - U_w U_w'. \tag{3.9}$$

Discretizing the left-hand side of (3.9) so that $\delta X P_i' = P_i - P_{i-1}$ where $\delta X = X_i - X_{i-1}, P_i = P(X_i)$, and adding to (3.8) we have, as the additional boundary condition (3.4) at the current station $X = X_i$

$$\delta X\left(\left.\frac{\partial^3 \Psi}{\partial Y^3}\right|_{Y=0} - V_w \left.\frac{\partial^2 \Psi}{\partial Y^2}\right|_{Y=0} - U_w U_w'\right) + \left.\frac{\partial^2 \Psi}{\partial X \partial Y}\right|_{Y=Y_\infty} + \delta X P_{i-1} = 0. \tag{3.10}$$

When the solution at $X = X_i$ has been determined using (3.1), (3.2), (3.3) and (3.10) the pressure P_i may be determined directly from either of (3.8) or (3.9).

As discussed by Riley [7] the case of oblique slot injection in supersonic flow provides examples of both a compressive and expansive free interaction in $X < 0$ depending upon the channel angle. In this paper we discuss examples of only the former type. The free interaction in $X < 0$ is initiated as in [7]. Thus at $X = -11.0$ we set $P = 10^{-4}$ and when the numerical integration is carried out in the manner described above we have $P(0) = 0.4624$ and the shear stress $\tau_w = \partial^2\Psi/\partial Y^2|_{Y=0} = 0.4794$. In the calculations for the supersonic case we have set $Y_\infty = 10$, $N = 34$ and $\delta X = 0.1$. The free interaction in $X < 0$ is calculated first so fixing $P(0)$. This implies that the flow in $X < 0$ has already anticipated a disturbance of a certain magnitude and this in turn implies that, for a given channel inclination, not all of the parameters W_s, λ and ℓ_s in (2.7) may be prescribed arbitrarily. In our calculations we have fixed the channel inclination and width at the same values as were used by Riley [7] in the example of a compressive free interaction discussed there. We have then, for each value of λ selected for (2.7), varied the value of the constant W_s until the downstream condition $P \to 0$ as $X \to \infty$ is satisfied. An incorrect choice for W_s will lead to $P \to \pm\infty$ as $X \to \infty$. By continuing our numerical integration up to a suitably large value of X (which we chose as $X = 16$) it is possible to adjust W_s until it is within some prescribed tolerance of its exact value. This is similar to the procedure adopted by Smith and Stewartson [5]. At the edge of the slot the solution of (3.1) exhibits a double structure with an inner boundary layer of thickness $O(X^{1/3})$ as $X \to 0$. This double structure is very similar in nature to that discussed in [5] and has not been specifically incorporated into our solution. The accuracy of our calculations can be estimated, in part at least, from the coefficients a_i in (3.6) where none of the last few has exceeded 10^{-9} in the course of the calculation. The results we have obtained for various values of λ are discussed in the next section.

(ii) *Subsonic flow*

The treatment for subsonic flow is inevitably different from that for supersonic flow on account of the differences between the pressure/displacement function relationships in (2.6a), (2.6b).

Different approaches to subsonic triple-deck problems have been introduced in the literature. Thus Jobe and Burggraf [8], in their study of the flow at the trailing-edge of a flat plate, use an overall iterative scheme in which for a given $A(X)$ the pressure field $P(X)$ is updated step-by-step as the lower deck equations are integrated. At the end of a complete sweep through the field of integration a new estimate of the displacement function is made from the inverse of the Hilbert integral (2.6b). The process is repeated, with a substantial degree of under-relaxation between the old and new values of $A(X)$, until the old and new values of $A(X)$ differ by an amount which is less than some prescribed tolerance, and overall convergence is achieved. By contrast Napolitano and Messick [6] in their study of uniform normal slot injection into a laminar boundary layer adopt the numerical technique of Napolitano, Werle and Davis [9] in which the problem is treated as an unsteady one. Thus a relaxation-like time derivative of A is added to the right-hand side of (2.3a) the solution of which is advanced in time using a two-sweep alternating direction implicit method until a convergence criterion is achieved.

In this paper we treat the lower-deck subsonic problem using the recently-introduced method of Veldman [10]. We believe that this method, which has still to be widely exploited, is the best available for problems of this type. As we shall see below it is relatively easy to implement and has proved to be completely stable in practice to the extent that over-relaxation is possible to accelerate the overall convergence.

Following Veldman we may first write, since $A(X) \to 0$ as $X \to \infty$, the Hilbert integral (2.6b) as

$$\pi P(X) = \int_{-\infty}^{\infty} A''(\xi) \log|\xi - X| \, d\xi. \tag{3.11}$$

If X_i is the current station (with i running from 1 to n) consider the contribution to the integral over an interval of length δX centred upon $\xi = \xi_j$. This may be approximated numerically as

$$(A_{j+1} - 2A_j + A_{j-1}) I_{i,j} / 2\delta X, \tag{3.12}$$

where

$$I_{i,j} = \log(|\xi_{j+\frac{1}{2}} - X_i||\xi_{j-\frac{1}{2}} - X_i|), \tag{3.13}$$

and $A_j = A(\xi_j)$, $\xi_{j+\frac{1}{2}} = \xi_j + \frac{1}{2}\delta X$ etc. Thus in this numerical approximation we may write

$$\pi P_i = A_s + F_{i,j} + A_i G_{i,i}, \tag{3.14}$$

where

$$\begin{aligned} G_{i,j} &= (I_{i,j+1} - 2I_{i,j} + I_{i,j-1})/2\delta X, \\ F_{i,j} &= \sum_{j=2}^{i-1} A_j G_{i,j} + \sum_{j=i+1}^{n-1} A_j G_{i,j}, \\ A_s &= \left(\int_{-\infty}^{X_{-\frac{1}{2}}} + \int_{X_{n+\frac{1}{2}}}^{\infty} \right) A''(\xi) \log|\xi - X_i| \, d\xi + \\ &\quad + \{(A_0 - 2A_1) I_{i,1} + A_1 I_{i,2} + A_n I_{i,n-1} + (A_{n+1} - 2A_n) I_{i,n}\}/2\delta X. \end{aligned} \tag{3.15a,b,c}$$

It is assumed that $A(X)$ for $X \leqslant X_1, X \geqslant X_n$ may be estimated from its asymptotic form so that A_s in (3.15c) may be evaluated.

Consider next the boundary condition (3.4) which we construct in this case at $X = X_i$ as follows. As before we have a relationship at $Y = 0$ between P and Ψ derived from (2.3), (2.4) as in (3.9). Equation (2.5b) gives us

$$\left.\frac{\partial \Psi}{\partial Y}\right|_{Y=Y_\infty} = Y_\infty + A_i, \tag{3.15d}$$

and we finally eliminate A_i, P_i from equations (3.9), (3.14) and (3.15d) to give the boundary condition (3.4) in the form

$$\delta X \left(\left.\frac{\partial^3 \Psi}{\partial Y^3}\right|_{Y=0} - V_w \left.\frac{\partial^2 \Psi}{\partial Y^2}\right|_{Y=0} - U_w U'_w \right) + \frac{G_{i,i}}{\pi} \left(Y_\infty - \left.\frac{\partial \Psi}{\partial Y}\right|_{Y=Y_\infty} \right) + + P_{i-1} - \frac{A_s + F_{i,j}}{\pi} = 0. \tag{3.16}$$

We are now in a position to determine $\Psi(X_i, Y)$ by our numerical procedure from (3.1), (3.2), (3.3) and (3.16). With the solution so determined we can now calculate A_i and P_i from (3.15d) and (3.14) respectively. In fact it proves convenient to work not with $F_{i,j}$ in (3.16) but, in order to accelerate convergence, with

$$\bar{F}_{i,j} = \sum_{j=2}^{i-1} \bar{A}_j G_{i,j} + \sum_{j=i+1}^{n-1} A_j G_{i,j}, \tag{3.17}$$

where $\bar{A}_j$ is the updated value calculated in the manner described above. Thus at the end of each iterative sweep we have a new set of values for A_i, P_i.

In order to implement the scheme outlined we need further information about the asymptotic properties of $A(X)$, $P(X)$ so that we may both initiate the calculation, and evaluate the quantity A_s in (3.16). As $X \to \pm\infty$ we have $A, P \to 0$ as the flow in the lower-deck region assumes a state of uniform shear. We consider how the flow returns to this state as $X \to +\infty$ and infer corresponding results as $X \to -\infty$ from the fact that $A' + iP$ is an analytic function of $X + iY$. We write, as $X \to \infty$,

$$\Psi \sim \tfrac{1}{2}\eta^2 X^{2/3} + \sum_{n=2}^{\infty} X^{(2-n)/3} F_n(\eta), \qquad \eta = Y/X^{1/3}, \tag{3.18}$$

and

$$P \sim \sum_{n=1}^{\infty} P_n X^{-n/3}. \tag{3.19}$$

The constants P_1 , P_2 = 0 for otherwise the upstream conditions are violated. The function $F_2(\eta)$ then satisfies

$$F_2''' + \tfrac{1}{3}\eta^2 F_2'' = 0, \tag{3.20}$$

$$F_2(0) = \Psi(\ell, 0), \quad F_2'(0) = 0, \quad F_2'' \to 0 \quad \text{as} \quad \eta \to \infty,$$

where

$$\Psi(\ell, 0) = -\int_0^{\ell} V_w(X)\,\mathrm{d}X.$$

The solution of (3.20) is

$$F_2(\eta) = \Psi(\ell, 0) + C\eta \int_0^{\eta} e^{-\frac{1}{9}t^3}\,\mathrm{d}t - C\int_0^{\eta} te^{-\frac{1}{9}t^3}\,\mathrm{d}t, \tag{3.21}$$

where the constant C is as yet undetermined. Smith and Stewartson [5] use a mass balance argument to determine C but a more direct method has been suggested by Watson*. Thus, from (2.3), (2.5b) we may show that as $Y \to \infty$

$$\Psi \sim \tfrac{1}{2} Y^2 + YA(X) + \tfrac{1}{2}A^2(X) + P(X), \tag{3.22}$$

and since $A, P \to 0$ as $X \to \infty$ we have, comparing (3.18), (3.21) with (3.22)

* E. J. Watson, private communication

$$C = \frac{\Psi(\ell, 0)}{3^{1/3}\,\Gamma(2/3)},$$

and then

$$A(X) \sim \frac{\Psi(\ell, 0)\,\Gamma(1/3)}{3^{2/3}\,\Gamma(2/3)}\, X^{-1/3}$$

$$\sim \beta_2\, X^{-1/3}, \quad \text{say}, \tag{3.23}$$

where $\beta_2 = 0.951\ \Psi(\ell, 0)$. The result (3.23) implies that the displacement function has the asymptotic form $A \sim \bar{\beta}_2 (-X)^{-1/3}$ as $X \to -\infty$. However the perturbation of the oncoming uniform shear which reflects such behaviour yields a perturbation solution which is unbounded, and as a consequence P_4 in (3.19) is chosen so that $\bar{\beta}_2 \equiv 0$, thus

$$P_4 = 0.183\ \Psi(\ell, 0). \tag{3.24}$$

The constant P_3 is, as yet, undetermined. However if we are to satisfy the conditions $A \to 0$ as $X \to \pm\infty$ then it follows that $P_3 \equiv 0$.

As $X \to -\infty$ we now have, from (3.23), (3.24), $P(X) \sim -0.366\ \Psi(\ell, 0)(-X)^{-4/3}$; this suggests that as $X \to -\infty$ we write

$$\Psi \sim \tfrac{1}{2} Y^2 - 0.488\ \Psi(\ell, 0)(-X)^{-4/3}\phi(\eta), \quad \eta = Y/(-X)^{1/3}, \tag{3.25}$$

where $\phi(\eta)$ satisfies

$$\phi''' - \tfrac{1}{3}\eta^2\phi'' - \tfrac{4}{3}\eta\phi' + \tfrac{4}{3}\phi = 1,$$
$$\phi(0) = \phi'(0) = \phi''(\infty) = 0. \tag{3.26}$$

The solution of (3.26) gives, with (2.5b) and (3.25)

$$A(X) \sim 0.476\ \Psi(\ell, 0)(-X)^{-5/3} \tag{3.27}$$

as $X \to -\infty$.

We may now commence our numerical calculation at a sufficiently large negative value of X using (3.25). In the step-by-step integration the results (3.23) and (3.27) are now used to estimate the quantity A_s in the condition (3.16). In all our calculations we have worked in the range $-7 \leqslant X \leqslant 23$, and if to start our solution procedure for a particular pair (λ, W_s) in (2.7) we assume initially that $A(X) \equiv 0$, about 15 iterative sweeps are required to ensure that $|A^{(n+1)} - A^{(n)}| < 10^{-4}$, where the superscript denotes the iteration, when no over-relaxation is employed. However Veldman [10] suggests that over-relaxation is possible with his method, and indeed we have found that if overall over-relaxation on $A(X)$, with an over-relaxation parameter 1·2, is employed then the number of iterations to achieve the same accuracy is reduced to 9 or 10. With the above tolerance on A we have, correspondingly, $|P^{(n+1)} - P^{(n)}| < 10^{-3}$ and

$|\tau_w^{(n+1)} - \tau_w^{(n)}| < 10^{-3}$. For this case of subsonic flow we have carried out various numerical experiments, from which we note, first of all, that it has been necessary to take $Y_\infty = 20$. For the first set of calculations we have chosen $N = 34$, $\delta X = 0.2$; none of the last nine coefficients a_i in (3.6) exceeds 10^{-7}. For $\lambda = 5$ we have repeated the calculation with $N = 24$, $\delta X = 0.2$, and we have found that the converged values for P and τ_w change nowhere by an amount exceeding $\frac{1}{2}$%. This suggests that the representation (3.6) (for which the last coefficient never exceeded 10^{-6}) with $N = 24$ is adequate to resolve the boundary-layer structure. Our final set of calculations, and it is the results from these that are presented here, were carried out with $N = 24$, $\delta X = 0.1$. The largest difference which we have noted between these and the first set is, for the pressure, 13 units in the third decimal place at $X = 0.4$ for $\lambda = 5$, which represents a difference of about 4% there, and for τ_w, 15 units in the third decimal place at $X = 4.0$ for $\lambda = 5$ which represents a difference of 3% there. Elsewhere the difference between the two sets of solutions is much less. We therefore suggest that the solutions presented here are accurate to $O(10^{-3})$.

4. Results

We have confined the calculations in this paper to the configuration shown in Fig. 1 for the following reason. Riley [7] in his study of oblique slot blowing with constant U_w, V_w considered supersonic flow only with a particular value of the pressure at the edge of the slot $X = 0$ following a free interaction in $X < 0$; the values of U_w, V_w were fixed and the length of the slot, ℓ, which is required if the downstream boundary condition is to be satisfied was determined. In our calculations we have fixed $\ell = 5.563$ with the angle of inclination of the channel $\theta = \tan^{-1} 3$, in our scaled co-ordinates, corresponding to one of Riley's examples in which $U_w = 0.2$, $V_w = 0.6$. This configuration is shown in Fig. 1 together with channel velocity profiles corresponding to different values of λ in (2.7), the case $\lambda = \infty$ yields constant $U_w = 0.2$ and $V_w = 0.6$. Although the results which we have obtained are qualitatively similar in both the supersonic and subsonic cases we discuss each separately.

(i) *Supersonic flow*

As we have already mentioned the free interaction which we have induced in $X < 0$ gives a pressure rise such that $P(0) = 0.4624$ as in Riley [7]. For the case of uniform, oblique injection Riley showed that at the edges of the slot the pressure gradient is infinite and the shear stress is also discontinuous and unbounded. For the case of uniform normal injection studied by Smith and Stewartson [5] there is a finite discontinuity in the pressure gradient at the slot edges where the shear stress is continuous. In the present case the effect of the non-uniform injection rate, for which the velocity at $Y = 0$ is continuous as in (2.7), is to make the pressure gradient continuous at the slot edges where the shear stress remains finite. We have carried out the calculations in the manner described in Section 3, with a terminal point $X = 16.0$, for three values of the parameter λ in (2.7) namely $\lambda = 1, 3, 5$ which, as can be seen from Figure 1, enables us to model a channel flow that varies from the fully-developed type to that in which there is a clearly defined core flanked by boundary layers. The results obtained are not unexpected.

Consider first the pressure distribution for $\lambda = 1$ shown in Fig. 2. As may be seen from equation (2.6a), the pressure is influenced by local conditions and not conditions downstream. Thus when the velocities in the slot are small, as they are initially, the solution is dominated by the free interaction; after a relatively short distance, as the oncoming stream feels the full effect of the injection, the pressure falls across the slot and then beyond it gradually assumes its free-stream value. As λ increases the effect of the free interaction close to the leading edge of the slot diminishes, and the pressure distribution approaches that for the case $\lambda = \infty$ which we have taken from [7].

The displacement function $A(X)$ corresponding to these pressure distributions is shown in Fig. 3. We see that A decreases in the free-interaction region, continues to fall across the slot

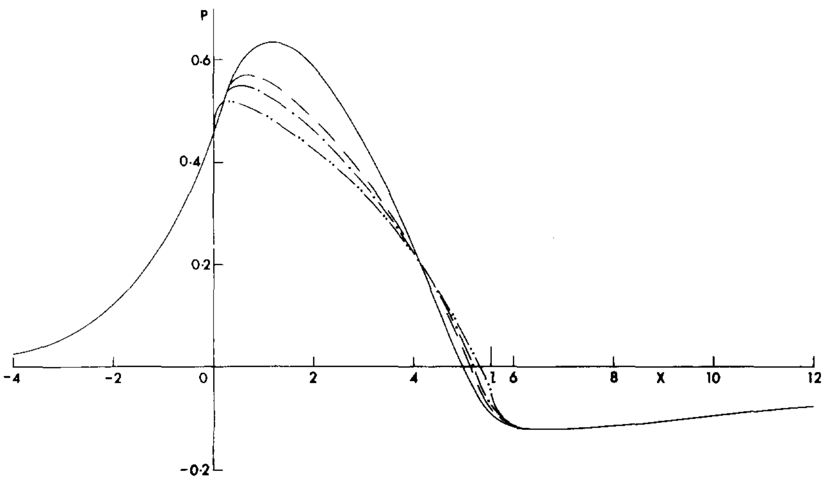

Figure 2. The pressure distribution $P(X)$ in the supersonic case.
——— $\lambda = 1$, – – – – $\lambda = 3$,
–·–·– $\lambda = 5$, –··–··– $\lambda = \infty$.

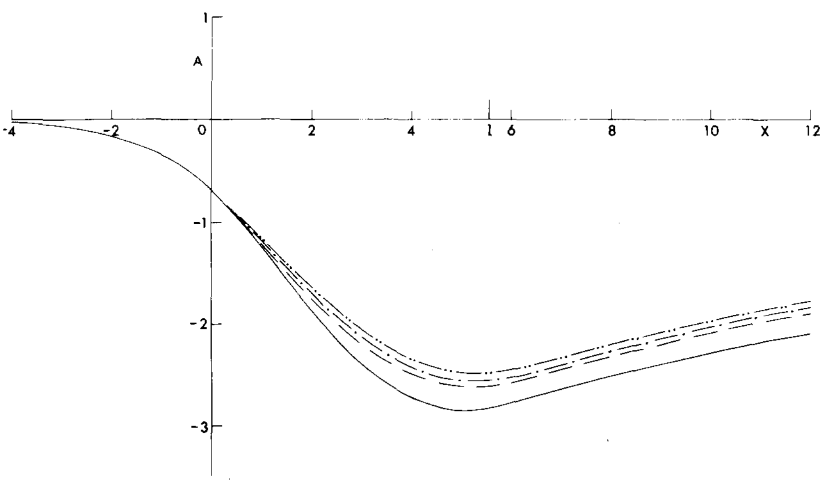

Figure 3. The displacement function $A(X)$ in the supersonic case. The different cases correspond to those in Figure 2.

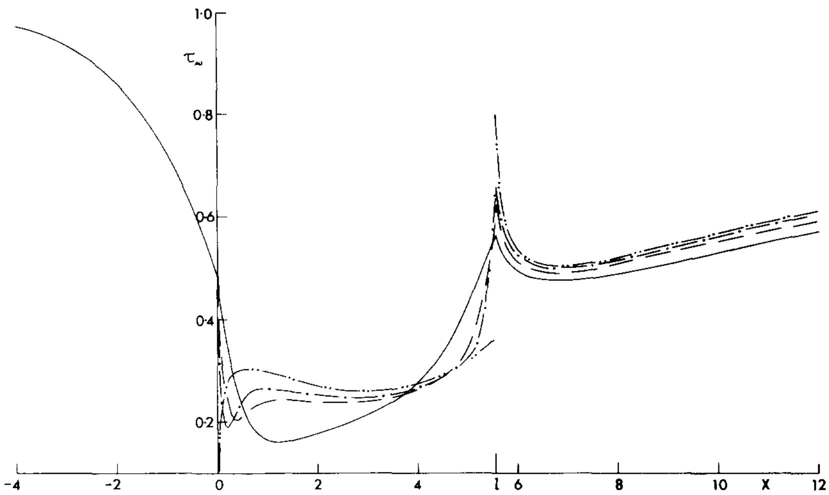

Figure 4. The wall shear stress $\tau_w(X)$ in the supersonic case. The different cases correspond to those in Figure 2.

and finally beyond the slot gradually assumes its undisturbed value as in equations (3.23). Again as λ increases the solution for A approaches that for $\lambda = \infty$ which corresponds to uniform oblique injection.

In Fig. 4 we show the shear stress at $Y = 0$ for these cases. As λ increases the initial fall in shear stress at the leading edge of the slot increases, and in the formal limit $\lambda = \infty$ the shear stress is unbounded as in [7]. As in the case $\lambda = \infty$ the shear stress recovers, having passed through a clearly defined minimum, to reach a maximum at the slot trailing edge beyond which the undisturbed value is gradually approached. We see that as λ increases the distribution approaches that for uniform injection corresponding to $\lambda = \infty$.

(ii) *Subsonic flow*

In Figs 5 to 7 we show the corresponding results for subsonic flow except that in this case there is no solution available for uniform oblique injection ($\lambda = \infty$). Calculations for the case of uniform normal injection have been carried out by Napolitano and Messick [6] for a slot of a length which is comparable to ours, and for an injection velocity varying between 0.1 and 1.5. Their technique, as indicated earlier, is based upon the solution of a time-dependent problem. In their calculations they set $A = P = 0$ at the upstream location where the solution commences, and $dA/dX = 0$ at the terminal point. If (X_I, X_F) represent the initial and final stations respectively they adopt intervals (−20.8, 25.8) to (−35.2, 45.8), depending upon the value of V_w, with the corresponding value of Y_∞ varying between 12.0 and 21.0. A variable step-length $\delta X \geqslant 0.25$ was used in each calculation with a uniform normal step-length $\delta Y = 0.3$. In our calculations we have worked on the interval (−7.0, 23.0) with $Y_\infty = 20.0$, and for the results we present a fixed step-length $\delta X = 0.1$. In addition we have incorporated the leading term in the asymptotic expansions for A, P as $X \to \pm \infty$ into our solution procedure.

The pressure distributions shown in Fig. 5 are not dissimilar to those of the supersonic case in that, overall, the pressure is seen to fall across the slot. However since in the subsonic case the pressure responds not simply to local conditions but rather to the global situation, the pressure distribution is more symmetrically disposed across the slot than in the supersonic case.

The shear stress is shown in Fig. 6. For $X < 0$ the values of the shear stress are virtually indistinguishable for the different values of λ. For the supersonic case they are of course exactly the same resulting from the free interaction. For $X > 0$ the shear stress shows the same tendency to decrease sharply with increasing λ, as in the supersonic case, and reaches its maximum value across the slot at its trailing edge.

Finally we show, in Fig. 7, the displacement function $A(X)$ for the subsonic case which is seen to vary in a similar manner to that which we have calculated in the supersonic case.

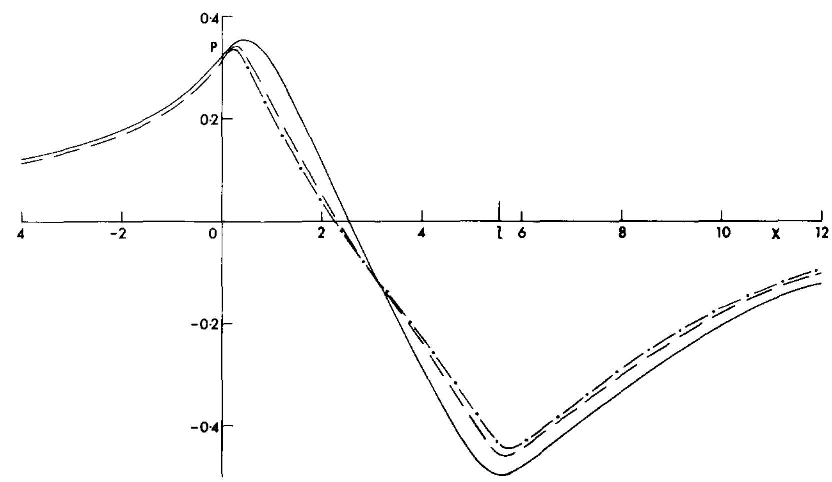

Figure 5. The pressure distribution $P(X)$ in the subsonic case for the pairs of values (W_s, λ). ——— (0.9789, 1), – – – – (0.7340, 3), –·–·– (0.6917, 5). For $X < 0$ the cases $\lambda = 3, 5$ are indistinguishable.

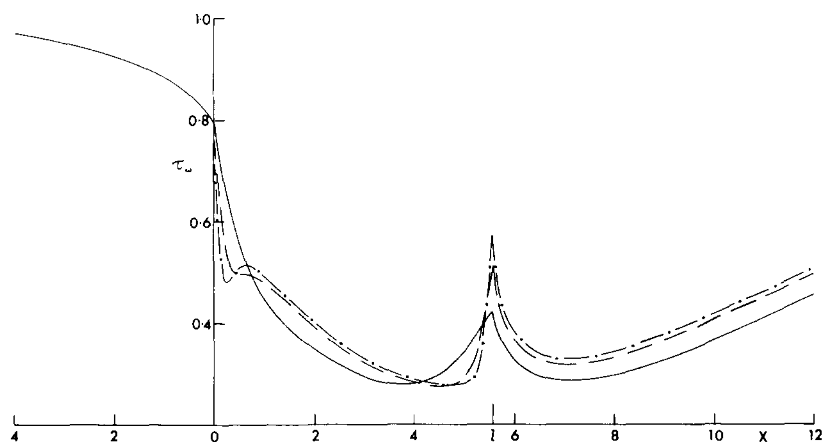

Figure 6. The wall shear stress $\tau_w(X)$ in the subsonic case. The different cases correspond to those in Figure 5. For $X < 0$ the three cases are indistinguishable.

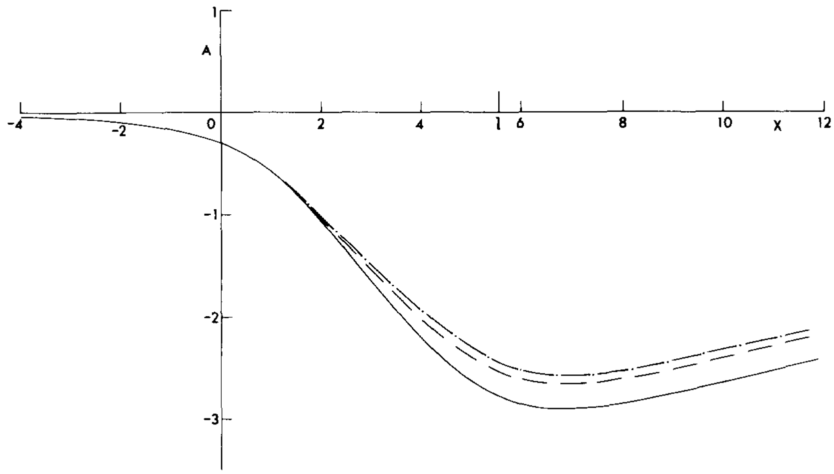

Figure 7. The displacement function in the subsonic case. The different cases correspond to those in Figure 5.

5. Discussion

In this paper we have considered the behaviour of a boundary-layer flow into which there is injection from a slot in the boundary. The slot width and injection rate are chosen so that the interaction between the boundary layer and outer inviscid flow which is provoked may be described by triple-deck theory. The work is therefore an extension of that to be found in [5], [6] and [7]. In contrast to classical boundary-layer theory in which, for example, the displacement effect of the boundary layer is determined following the determination of the pressure, there is no such hierarchy in triple-deck theory which requires the simultaneous calculation of pressure and displacement.

In the examples which we have considered the injection velocity at the boundary is represented by a non-uniform flow, which models the flow in a channel below the boundary from which the injected fluid emerges. This results in a more realistic injection velocity profile than that considered earlier. In particular the discontinuity in the pressure gradient at the edges of the slot is removed.

The calculations which we have carried out include examples of both supersonic and subsonic flow. In the former case the pressure-displacement relation is given by Ackeret's law, as in (2.6a), which depends upon local conditions. By contrast, in subsonic flow the pressure-displacement relation is given by the Hilbert integral in (2.6b) which reflects the global elliptic nature of the interaction in this case. The differences in the results which emerge are typified by the pressure distributions shown in Figs 2 and 5. Consider the case which, with reference to Fig. 1, corresponds to an almost fully-developed injection velocity profile. In each of the supersonic and subsonic cases there is a broad similarity in the results in the sense that the pressure rises ahead of the slot, falls across most of it, and finally recovers to its undisturbed value. There are, however, differences in detail. Consider the supersonic case shown in Fig. 2. There the pressure rise ahead of the slot is the result of a free interaction, and since in the slot the injection rate is initially small the free-interaction effect dominates and the pressure continues to rise. Ultimately the local velocity is strong enough to reverse this trend. By contrast, in the subsonic case shown in Fig. 5, the pressure at any point is sensitive to flow conditions everywhere. A consequence of this is that not only does the pressure disturbance penetrate further upstream compared with the supersonic case, but it responds more readily to the slot conditions reaching a lower maximum more quickly and falling to a much lower minimum. Also we note that as λ increases the pressure profiles in the subsonic case bear a closer resemblance to one another, and are more nearly symmetrical across the slot, than in the supersonic case. We interpret this as a clear indication that in the subsonic case the pressure at each point is reacting to the overall disturbance provided by the slot injection which it cannot do in the supersonic case.

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